Algebra 3 Midterm

Write all your reasoning carefully and precisely. Unless otherwise stated, all rings are commutative with a 1 that is different from 0. Q **is the ring of rational numbers and** C **the ring of complex numbers. Total points 10 × 5 = 50.**

1. a) For a ring R with ideals I and J, show that the following two statements are equivalent.

(i) The canonical map from R to (R/I) × (R/J) is an isomorphism.
(ii) I + J = R and I ∩ J = 0.

b) Show by an example that it is possible for R and (R/I) × (R/J) to be isomorphic without the conditions in part (ii) being true.

2. a) Define the annihilator of an element b in a ring R by ann(b) = { a | ab = 0 }. Show briefly that ann(b) is an ideal of R.

b) Now let R be of cardinality mn where m and n are relatively prime natural numbers. Show that R is isomorphic to the product of two rings, one of cardinality m and the other of cardinality n. You may want to consider the ideals ann(m.1) and ann(n.1). (Recall that m.1 denotes 1 in R added m times.)

c) For a squarefree natural number s, show that up to isomorphism, there is a unique ring of cardinality s.

3. Consider the homomorphism ϕ: C[X,Y] → C[t] between polynomial rings taking any polynomial f(X,Y) to $f(t^2, t^3)$.

a) Show that the kernel of ϕ is a principal ideal and find its generator.
b) If in the definition of the map, $f(t^2, t^3)$ is replaced by $f(t^m, t^n)$ for positive integers m and n, how would your answer to part a change?

4. Consider a ring homomorphism φ from a ring R to a ring S and an ideal I of S.

a) Show that if I is a prime ideal of S then $\varphi^{-1}(I)$ is a prime ideal of R.
b) Show that if I is a maximal ideal of S then $\varphi^{-1}(I)$ need not be a maximal ideal of R but it is a prime ideal of R.
c) Show that for the homomorphism in problem 3, inverse image of every maximal ideal in C[t] is a maximal ideal in C[X,Y]. State with reason precisely which maximal ideals of C[X,Y] arise as such inverse images. You may find it helpful to think geometrically.

5. In this question we want to find all ring homomorphisms between given rings.

a) Show that inclusion is the only homomorphism from Q to C.
b) Find all homomorphisms from Q[x] / $(x^3 + 4x^2)$ to C. How many homomorphisms are there if $x^3 + 4x^2$ is replaced by an arbitrary polynomial f(x)?
c) Describe all homomorphisms from Q(x), i.e., the ring of rational functions with rational coefficients, to C.
d) For the ring R of real numbers, show that the only homomorphism from R to R is the identity. (Hint: consider positive reals.)

Turn over for extra credit problems→

Extra credit problems

A. True or false? For any ring D, there are only finitely many ring homomorphisms from the ring Q[x] / (f(x)) to D, where f is a fixed polynomial with rational coefficients.

B. Let R be the ring of real numbers. Find all ring homomorphisms from R[[x]] to C. How about homomorphisms from Q[[x]] to C?

C. We want to find all rings (not necessarily commutative) of cardinality 2009. Show that up to isomorphism there are exactly 4 such rings.

D. For a commutative ring R, are the rings R[x][[y]] and R[[y]][x] isomorphic?